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A RELATIVE NASH THEOREM

BY

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ABSTRACT. We prove that if M is a closed smooth manifold and M_i, $i = 1, \ldots, k$, are transversally intersecting closed smooth submanifolds of M, then there exist a nonsingular algebraic set Z and nonsingular algebraic subsets Z_i, $i = 1, \ldots, k$, of Z such that $(M; M_1, \ldots, M_k)$ is diffeomorphic to $(Z; Z_1, \ldots, Z_k)$. We discuss a generalization and the consequences of this result.

1. Introduction. In 1952 Nash proved that every smooth closed manifold is a *component of a nonsingular algebraic set*[1] [N]; and he conjectured that it can in fact be made a *nonsingular algebraic set*. In 1973 Tognoli verified this conjecture [T]. We will prove a "relative" version of this result which in particular implies the following:

THEOREM 1.1. *Let M be a closed smooth manifold, and let M_i, $i = 1, \ldots, k$ be closed smooth submanifolds of M in general position. Then there is a nonsingular algebraic set Z and nonsingular algebraic subsets Z_i, $i = 1, \ldots, k$, of Z and a diffeomorphism ϕ: $M \to Z$ with $\phi(M_i) = Z_i$, $i = 1, \ldots, k$.*

Our main motivation for proving this theorem was to prove a converse to the resolution theorem of [H]. The resolution theorem says that: if a closed smooth manifold Σ is a link of a real algebraic set Z with an isolated singularity (i.e. $Z \approx \text{cone}(\Sigma)$); then Σ bounds a compact smooth manifold W with the property that W has transversally intersecting closed smooth submanifolds M_i, $i = 1, \ldots, k$, with $W / \cup M_i \approx \text{cone}(\Sigma)$ (i.e. $\cup M_i$ is a spine of W).

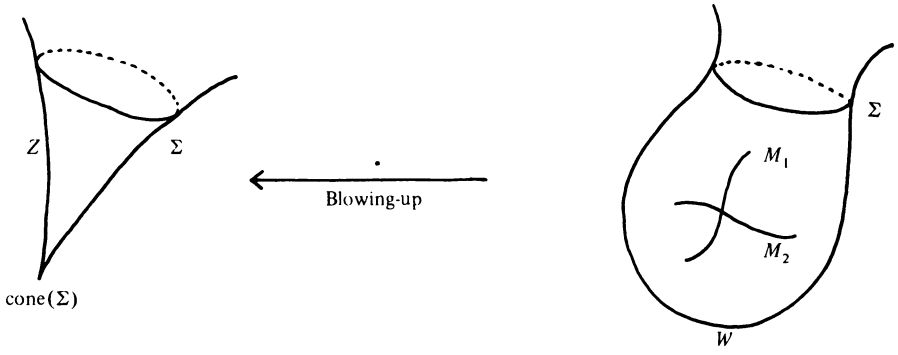

FIGURE 1

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[1] Here algebraic set means a zero set of real polynomial functions.

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W is obtained by successively blowing up the singular point of Z. Now by using Theorem 1.1 we can prove a converse to this. Namely, if we are given Σ bounding such a W; we let $M = W \cup_\partial -W$; and then by the theorem we can make $(M, M_1, \ldots, M_k)$ diffeomorphic to nonsingular algebraic sets $(Z, Z_1, \ldots, Z_k)$. We can then algebraically collapse (blow down) the algebraic subsets Z_i, $i = 1, \ldots, k$ of Z to a point. Then we get an algebraic set $Z/\cup Z_i \approx M/\cup M_i$. In particular this makes cone(Σ) an algebraic variety. This was outlined in [$\mathbf{AK}_1$]. We proved Theorem 1.1 back in 1976 for this purpose. But since then, we have found a quick proof of this theorem under the additional assumption that each M_i has codimension one [$\mathbf{AK}_2$]. We also proved that every bounding manifold Σ must bound a W which has a spine consisting of union of transversally intersecting closed *codimension-one* submanifolds M_i, $i = 1, \ldots, k$ [$\mathbf{AK}_2$], [$\mathbf{AK}_3$]. Hence in [$\mathbf{AK}_2$] we were able to prove the result "Σ is a link of an algebraic set with isolated singularity if and only if it bounds" without using Theorem 1.1. But Theorem 1.1 remains interesting because it gives many different ways of making cone(Σ) an algebraic variety with an isolated singularity. This theorem can be useful for "classifying" all the algebraic sets, with an isolated singularity and a given link Σ.

There is an old question whether smooth closed submanifolds of a nonsingular algebraic set are isotopic to nonsingular algebraic subsets. Our theorem says that this can be done if we are allowed to move the ambient algebraic set by an isotopy (to another nonsingular algebraic set). As an application of Theorem 1.1 we prove that the interior of a smooth compact manifold with boundary is diffeomorphic to an open nonsingular algebraic set, which is properly imbedded in $\mathbf{R}^n$ for some n (Corollary 2.11).

In §2 we give some definitions and lemmas. We often refer to [$\mathbf{AK}_2$] for proofs of some of these basic lemmas. We also state the main result (Theorem 2.10) in this section. In §3, we give a quick proof of a weaker version of Theorem 1.1. We recommend that the reader should read this section before reading the main proof in §4. In §2 we make a conjecture which roughly says that up to diffeomorphism $\mathbf{Z}/2\mathbf{Z}$ homology classes of a nonsingular algebraic set are represented by algebraic subsets; if this conjecture turns out to be true then the proof in §3 can be considered as the main proof. §3 is a good warm-up exercise for §4. Other than Corollary 2.11 the reader can mostly skip §2 and use it as a reference section for definitions and lemmas used in the other sections.

2. Preliminary lemmas.

DEFINITION. A *real algebraic set* is a set V of the form $V(I) = \{x \in \mathbf{R}^n | p(x) = 0,$ for all $p \in I\}$ where I is a set of polynomial functions from $\mathbf{R}^n$ to $\mathbf{R}$.

In this paper "algebraic set" will always mean "real algebraic set".

DEFINITION. A point x in an algebraic set $V \subset \mathbf{R}^k$ is called *nonsingular of dimension d in V* if there are polynomials $p_i(x)$, $i = 1, \ldots, k - d$, $x \in \mathbf{R}^n$ and a neighborhood U of x in $\mathbf{R}^k$ so that $p_i(V) = 0$ and

(i) $V \cap U = U \cap \bigcap_{i=1}^{k-d} p_i^{-1}(0)$,

(ii) the gradients $(\nabla p_i)_x$, $i = 1, \ldots, k - d$, are linearly independent on U.

We define dim $V = \max\{d \mid$ there is an $x \in V$ which is nonsingular of dimension $d\}$.

DEFINITION. For an algebraic set V, Nonsing $V = \{x \in V \mid x$ is nonsingular of dim $V\}$ and Sing $V = V -$ Nonsing V. We say that an algebraic set V is nonsingular if Sing $V = \varnothing$.

Given a polynomial p: $\mathbf{R}^n \to \mathbf{R}$ of degree d we associate a homogeneous polynomial p^*: $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ by $p^*(t, x) = t^d p(x/t) = \sum_{i=0}^{d} t^{d-i} p_i(x)$ where p_i is the sum of the monomials of p of degree i. Let θ: $\mathbf{R}^n \to \mathbf{R}P^n$ be the imbedding $\theta(x_1, \ldots, x_n) = (1 : x_1 : x_2 : \cdots : x_n)$, and let $V(p^*) \subset \mathbf{R}P^n$ be the projective algebraic set given by the zeros of p^*. Then $V(p^*) \cap \theta(\mathbf{R}^n) = \theta(p^{-1}(0))$. We say that the polynomial $p(x)$ is *overt* if $V(p^*) = \theta(p^{-1}(0))$. This means that $p(x)$ does not "hide" zeros at infinity, or more precisely $p_d^{-1}(0)$ is $\{0\}$ or empty.

DEFINITION. An algebraic set V is called *projectively closed* if $V = p^{-1}(0)$ for some overt polynomial $p(x)$.

Hence an algebraic set V is projectively closed if and only if $\theta(V)$ is a projective algebraic subset of $\mathbf{R}P^n$.

DEFINITION. Let V and W be algebraic sets and let U be an algebraic subset of V. Let $W \subset \mathbf{R}^n$. Then a function f: $V - U \to W$ is called a *rational function* if there are polynomials p: $V \to \mathbf{R}^n$ and q: $V \to \mathbf{R}$ so that $q^{-1}(0) \subset U$ and $f(x) = p(x)/q(x)$ for all $x \in V - U$. If U is empty then we call f an *entire rational function*.

DEFINITION. Let W be a smooth manifold. We say that a finite collection of closed submanifolds $M_\alpha \subset W$, $\alpha \in A$ are in *general position* if for each $\alpha \in A$ and $A' \subset A - \alpha$, M_α intersects $\bigcap_{\alpha' \in A'} M_{\alpha'}$ transversally.

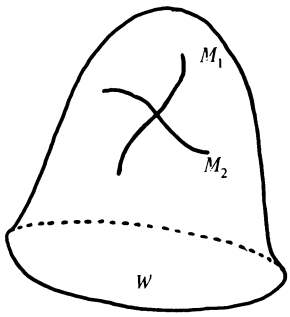

FIGURE 2

Next we state a few basic lemmas; they are useful for the main proofs. Lemmas 2.1, 2.2, 2.3, 2.4, 2.5, 2.6 correspond to Lemmas 2.9, 1.4, 1.6, 1.7, 2.1, 2.2 of [$\mathbf{AK}_2$] respectively. Hence we refer the reader to [$\mathbf{AK}_2$] for the proofs of these lemmas, except Lemma 2.4, which we prove here.

LEMMA 2.1. *Let $M_\alpha \subset W$, $\alpha \in A$ be a finite collection of submanifolds in general position and h_t^α: $M_\alpha \to W$, $t \in [0, 1]$, $\alpha \in A$ are small isotopies of imbeddings with $h_0^\alpha =$ inclusion. Then there is an isotopy h_t: $W \to W$ so that $h_t(M_\alpha) = h_t^\alpha(M_\alpha)$ for each $t \in [0, 1]$ and $\alpha \in A$ and h_0 is the identity. h_t can be arbitrarily small if the h_t^α are arbitrarily small.*

REMARK. The isotopy h_t covers h_t^α setwise, not pointwise.

LEMMA 2.2. *If U, V, W and X are algebraic sets with $X \subset V$ and $U \subset W$ and if $f\colon V - X \to W$ is a rational function, then $X \cup f^{-1}(U)$ is an algebraic set. Furthermore if $z \in f^{-1}(U)$ is nonsingular of dimension r in V, and $f(z)$ is nonsingular of dimension s in U and nonsingular of dimension t in W, and f is transverse to U at z, then z is nonsingular of dimension $r + s - t$ in $f^{-1}(U)$.*

LEMMA 2.3. *Let $V \subset \mathbf{R}^n$ be an algebraic set and $W \subset$ Nonsing V be an algebraic subset with* dim W = dim V. *Then $V - W \subset \mathbf{R}^n$ is an algebraic set. Also* Sing$(V - W)$ = Sing V *unless* W = Nonsing V.

LEMMA 2.4. *Any compact algebraic set V is the zero set of a proper polynomial (proper means that the inverse images of compacta are compact).*

PROOF. Let V be a compact algebraic set. Let $p\colon \mathbf{R}^n \to \mathbf{R}$ a polynomial with $V = p^{-1}(0)$. Assume p is not constant, otherwise the lemma is trivial. Let $B = \{x \in \mathbf{R}^n \mid |x| \leqslant R\}$; by taking R large enough we can assume that $V \subset \text{int}(B)$. Define $\theta\colon \mathbf{R}^n - \{0\} \to \mathbf{R}^n - \{0\}$ by $\theta(x) = x/|x|^2$. Note that $\theta^{-1} = \theta$. Let $q(x)$ be the polynomial $q(x) = p^*(|x|^2, x)$. Then since $p^{-1}(0)$ is compact,

$$q^{-1}(0) = \{0\} \cup \theta(p^{-1}(0) - \{0\}).$$

By the Lojasiewicz inequality [**L**] there are positive numbers b and a so that $|q(x)| \geqslant a|x|^b$ for all $x \in \theta(\mathbf{R}^n - B)$. Let degree$(p) = e$, then $q(x) = |x|^{2e}p(\theta(x))$. Hence for all $x \in \mathbf{R}^n - B$

$$\begin{aligned} |p(x)| &= |\theta(x)|^{-2e}|q(\theta(x))| \geqslant |\theta(x)|^{-2e}a|\theta(x)|^b \\ &= a|\theta(x)|^{b-2e} = a|x|^{2e-b}. \end{aligned}$$

Hence $|p(x)| \geqslant a|x|^d$ for all $x \in \mathbf{R}^n - B$ where d is some number, a is a positive number. Then $p(x)(1 + |x|^2)^{|d|+1}$ is a proper polynomial whose zero set is V.

LEMMA 2.5. *Let $K \subset \mathbf{R}^n$ be a compact set and $L \subset K$ be an algebraic set. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a smooth function so that for each p in L there is an algebraic set M containing L with $f(x) = 0$ for all x in M near p, and p is a nonsingular point of dimension d in M for some d. Then there is a polynomial $h\colon (\mathbf{R}^n, L) \to (\mathbf{R}, 0)$ which is arbitrarily close to f near K (C^r close for a given r). In addition if f has compact support to begin with, then f can be approximated by an entire rational function $u\colon (\mathbf{R}^n, L) \to (\mathbf{R}, 0)$ on all of $\mathbf{R}^n$ (not just on a compact set K).*

LEMMA 2.6. *Let W be an algebraic set and M be a codimension* 1 *closed smooth submanifold of* Nonsing W *which compactly separates W (this means that M separates W into two components and one of these components is compact). Also let L, V be algebraic sets with $L \subset$ Nonsing $V \subset M$. Then there is an arbitrarily small isotopy of M in* Nonsing W *which fixes L and carries M to a nonsingular algebraic set. If L is projectively closed this isotopy carries M to a nonsingular projectively closed set.*

Let $G(n, k)$ be the Grassmann manifold of k-planes in $\mathbf{R}^n$. Let $E(n, k) \to G(n, k)$ be the universal $\mathbf{R}^k$-bundle over $G(n, k)$. It is well known that $G(n, k)$ and $E(n, k)$ are nonsingular algebraic sets

$$G(n, k) = \{A \in \mathbf{R}^{n^2} \mid A^t = A, A^2 = A, \text{trace } A = k\},$$
$$E(n, k) = \{(A, y) \in G(n, k) \times \mathbf{R}^n \mid Ay = y\}$$

where we view the elements of $\mathbf{R}^{n^2}$ as $(n \times n)$-matrices. Every such matrix $A \in G(n, k)$ gives rise to a k-plane image(A) and conversely to every k-plane we associate the projection matrix onto this plane. Define

$$E^*(n, k) = \{(A, y, t) \in E(n, k) \times \mathbf{R} \mid |y|^2 + t^2 = t\}.$$

It can be checked that $E^*(n, k)$ is nonsingular and the projection $E^*(n, k) \to G(n, k)$ is the k-sphere bundle obtained from $E(n, k)$ by compactifying each fiber to a sphere.

LEMMA 2.7. *Let $L \subset V \subset \mathbf{R}^n$ be nonsingular algebraic sets. Then there is a rational function u: $L \to G(n, k)$ where u is the classifying map of the normal bundle $\nu(L, V)$ where k = codimension of L in V. In other words, $u(x)$ is the normal k-plane to L which lies in the tangent plane of V at x.*

PROOF. Let r = codimension of L in $\mathbf{R}^n$, and s = codimension of V in $\mathbf{R}^n$. It is known that the Gauss (classifying) maps α_L and α_V of the normal bundles of L and V in $\mathbf{R}^n$ are rational functions (see Lemma 2.3 of [$\mathbf{AK}_2$]). Denote $\alpha_V|_L$ by α_V

$$\alpha_L: L \to G(n, r), \quad \alpha_V: L \to G(n, s).$$

Then u: $L \to G(n, k)$ is given by $u = (I - \alpha_V)\alpha_L$ (this formula says that first project on the plane α_L then project onto the complementary plane to α_V in α_L), hence u is rational.

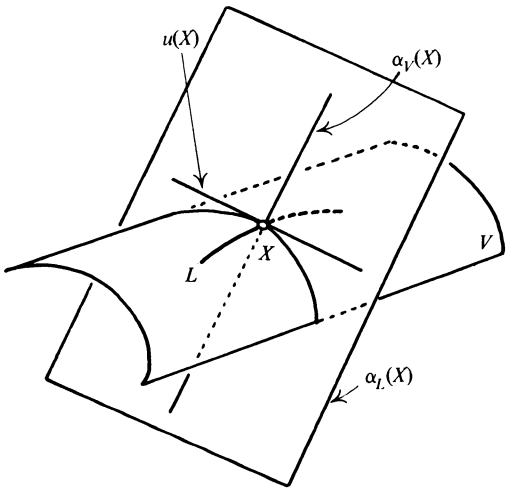

FIGURE 3

DEFINITION. An algebraic set X has *totally algebraic homology* if for every $\alpha \in H_k(X; \mathbf{Z}/2\mathbf{Z})$ there is a positive integer n and a nonsingular projectively closed k-dimensional algebraic subset $V \subset X \times \mathbf{R}^n$ so that $\pi_*([V]) = \alpha$ where $\pi\colon V \to X$ is the induced projection and $[V] \in H_k(V; \mathbf{Z}/2\mathbf{Z})$ is the fundamental class of V.

REMARK. The Schubert cycles are algebraic subsets of $G(n, k)$ which generate the $\mathbf{Z}/2\mathbf{Z}$-homology. The Schubert cycles are sometimes singular; but by the following remark $G(n, k)$ has totally algebraic homology (for more details see Lemma 2.7 of [$\mathbf{AK}_2$]).

In the above definition we can replace the requirements nonsingular and projectively closed by the requirement that V be compact. This can be seen as follows. Let V' be the smallest projective algebraic set containing V. Then $V' - V$ is a projective algebraic set, hence $\dim(V' - V) < \dim V = k$. Therefore $V' - V \subset \text{Sing } V'$. By resolving the singularities of V' [**H**] we get a nonsingular k-dimensional projective algebraic set $W \subset \mathbf{R}P^m$ and a rational function $r\colon W \to V'$ so that $r^{-1}(\text{Nonsing } V')$ is dense in W and $r|r^{-1}(\text{Nonsing } V')$ is a diffeomorphism onto Nonsing V'. Hence $r(W) \subset V$ and $r_*([W]) = [V']$. Let $\theta\colon \mathbf{R}P^m \to G(m + 1, 1)$ be the canonical isomorphism, the Grassmann variety $G(m + 1, 1)$ is a subvariety of $\mathbf{R}^{(m+1)^2}$ then $Z = \{(r(x), \theta(x)) \in V \times G(m + 1, 1) | \, x \in W\}$ is a nonsingular projectively closed algebraic subset of $X \times \mathbf{R}^n \times \mathbf{R}^{(m+1)^2}$ which gives the same homology class as $[V]$.

LEMMA 2.8. *An algebraic set X has totally algebraic homology if and only if every element of $\eta_*(X)$ (unoriented bordism group of X) is represented by some $\pi\colon V \to X$ where V is a nonsingular projectively closed algebraic subset of $X \times \mathbf{R}^n$ for some n and π is induced by projection.*

PROOF. Since $ev\colon \eta_*(X) \to H_*(X; \mathbf{Z}/2\mathbf{Z})$ is onto where $ev(M \xrightarrow{f} X) = f_*[M]$, one way this proof is trivial. Now assume that X has totally algebraic homology. It is known [**CF**] that $\eta_*(X)$ is generated by elements of the form

$$U_i \times V_j \xrightarrow{\pi_{ij}} X$$

with

$$\pi_{ij}\colon U_i \times V_j \to V_j \xrightarrow{\alpha_j} X$$

where $\{U_i\}$ generate $\eta_*(\text{point})$ and $\{ev(\alpha_j)\}$ generate $H_*(X; \mathbf{Z}/2\mathbf{Z})$. Hence we can assume that V_j is a nonsingular algebraic subset of $X \times \mathbf{R}^m$ for some m and α_j is induced by the projection. We can assume that U_i are nonsingular algebraic sets in $\mathbf{R}^n$ [**M**]. Then $V_j \times U_i \subset X \times \mathbf{R}^m \times \mathbf{R}^n$ and the projection $\pi_{ij}\colon V_j \times U_i \to X$ generate $\eta_*(X)$ as a group. By translating we can make all $V_i \times U_j$ disjoint; since the group operation is disjoint union we are done.

CONJECTURE. Every compact smooth manifold is diffeomorphic to a nonsingular algebraic set with totally algebraic homology.

If the above conjecture were true, it would be an extremely useful fact. For instance it would allow the simpler proof of our main result (Theorem 2.10) in §3.

LEMMA 2.9. *Let X and W be nonsingular algebraic sets, so X has totally algebraic homology. Let M be a closed smooth submanifold of X and let $f\colon M \to W$ be a smooth map. Also let $L \subset M$ be an algebraic set, the germ of M at L be the germ of a nonsingular algebraic set V, and the germ of f at L be the germ of an entire rational function $v\colon V \to W$. Then there is an s, an isotopy h_t of $X \times \mathbf{R}^s$, an algebraic set $Z \subset X \times \mathbf{R}^s$, and an entire rational function $r\colon Z \to W$ so that*

(1) *$h_1(M \times 0)$ is a nonsingular component of Z,*

(2) *$h_t(x, 0) = (x, 0)$ for all $x \in L$, $t \in [0, 1]$,*

(3) *$r(x, 0) = f(x)$ for all $x \in L$,*

(4) *h_t can be as small as we want, and $rh_1|_{M \times 0}$ can be as close an approximation to f as we want (for some choice of Z and r).*

LEMMA 2.9′. *Under the same hypothesis of Lemma 2.9 if in addition we assume that W has totally algebraic homology then in the conclusion we get that Z is a nonsingular algebraic set and $h_1(M \times 0) = Z$. Also along with (2), (3), (4) we get*

(5) *If L is projectively closed then Z is projectively closed.*

Lemmas 2.9 and 2.9′ roughly say that if X is a nonsingular algebraic set with totally algebraic homology (e.g. $\mathbf{R}^n$) then any closed smooth submanifold M of X can be made a nonsingular algebraic set in $X \times \mathbf{R}^s$ for some s; and also a smooth function from M to a nice algebraic set may simultaneously be approximated by a rational function. We give a proof of Lemma 2.9; the proof of Lemma 2.9′ is similar but somewhat harder than this proof. See Proposition 2.8 of [**AK**$_2$] for the proof of Lemma 2.9′.

PROOF OF LEMMA 2.9. Let X and W be nonsingular algebraic subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Let T be a tubular neighborhood of M in X; and let $g\colon T \to E(n, k)$ be the classifying map of the normal bundle of M in X, where k is the codimension of M in X. Then g has the properties

(a) g is transverse to $G(n, k)$,

(b) $g^{-1}(G(n, k)) = M$.

$E(n, k)$ is a nonsingular algebraic set in $\mathbf{R}^q$ for some q, and $G(n, k)$ is a nonsingular algebraic subset of $E(n, k)$.

Let N be a neighborhood of L in M; by Lemma 2.7 $g|_N$ equals a rational function u. We can extend f to T by composing f with projection from T to M; also recall that $f|_N$ equals a rational function v. By Lemma 2.5 we can approximate $(f - v) \times (g - u)$ by a polynomial $\phi'\colon (\mathbf{R}^n, L) \to (\mathbf{R}^m \times \mathbf{R}^q, 0)$ near M. Let $\phi = \phi' + (v, u)$; then ϕ approximates $f \times g$ near M. Also $\phi|_L = v \times u$.

Now define $S = \{(x, y) \in X \times \mathbf{R}^{m+q} | \phi(x) + y \in W \times E(n, k), y \text{ is normal to } W \times E(n, k) \text{ at } \phi(x) + y \text{ in } \mathbf{R}^{m+q}\}$. S is an algebraic set. To see this, let $\rho\colon \mathbf{R}^{m+q} \to \mathbf{R}$ be a polynomial with $\rho^{-1}(0) = W \times E(n, k)$, and by Lemma 2.7 there is a rational function $\delta\colon W \times E(n, k) \to G(m + q, r)$, $r =$ codimension of $W \times E(n, k)$, which is the normal bundle map of $W \times E(n, k) \subset \mathbf{R}^{m+q}$. By considering the elements of $G(m + q, r)$ as $(m + q) \times (m + q)$ projection matrices onto r planes, we can write $S = \{(x, y) \in X \times \mathbf{R}^{m+q} | \rho(\phi(x) + y) = 0, \delta(\phi(x) + y)\, y = y\}$ i.e. S is an algebraic subset of $X \times \mathbf{R}^{m+q}$.

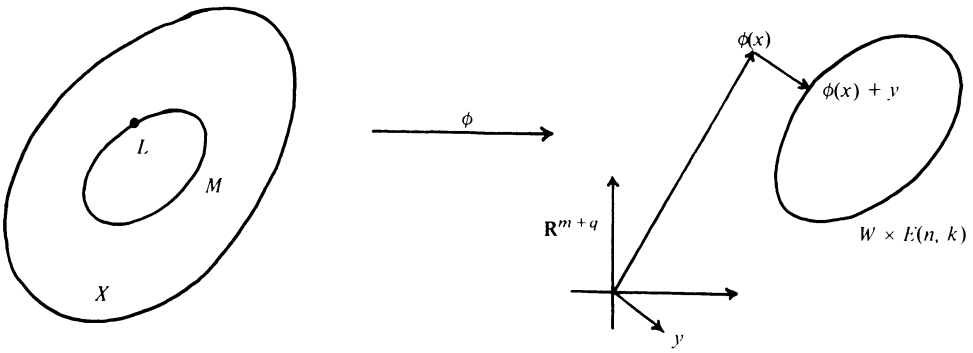

FIGURE 4

Let w: $S \to W \times E(n, k)$ be the rational function $w(x, y) = \phi(x) + y$. Since for every $x \in T$, $\phi(x)$ is close to $W \times E(n, k)$ there is a unique $y(x)$ so that $\phi(x) + y(x)$ is the unique closest point to $\phi(x)$ in $W \times E(n, k)$ (see Figure 4). Now define $J = \{(x, y(x)) \in S \cap (T \times \mathbf{R}^{m+q})\}$ then we have:

(A) J is an open subset of Nonsing(S).

(B) The function $x \to w(x, y(x))$ is close to $f \times g$ on T (since $y(x)$ is small).

(C) $w(x, 0) = (v(x), u(x)) = (f(x), g(x))$ for all $x \in L$.

Hence by transversality $(f \times g)^{-1}(W \times G(n, k)) = g^{-1}(G(n, k)) = M$ is isotopic to $\{x \in T | w(x, y(x)) \in W \times G(n, k)\}$ fixing L, which is in turn isotopic to $J \cap w^{-1}(W \times G(n, k))$ fixing $L \times 0$. Let

$$\hat{Z} = S \cap w^{-1}(W \times G(n, k)) \quad \text{and} \quad Z' = J \cap w^{-1}(W \times G(n, k)).$$

Then by Lemma 2.2 all the points of Z' are nonsingular of dimension dim M in $\hat{Z}$. Let $Z = \text{Sing}(\text{Sing}(\ldots (\text{Sing } \hat{Z}) \ldots))$ where we take Sing a number of times so that dim Z = dim M. Then notice that Z is an algebraic subset of $X \times \mathbf{R}^{m+q}$ and Z' is a nonsingular component of Z. We have constructed an isotopy h_t of $M \times 0$ to Z' which fixes $L \times 0$. To construct the rational function r, let π: $\mathbf{R}^m \times \mathbf{R}^q \to \mathbf{R}^m$ be the projection then let $r = \pi \circ w$. r has the required properties.

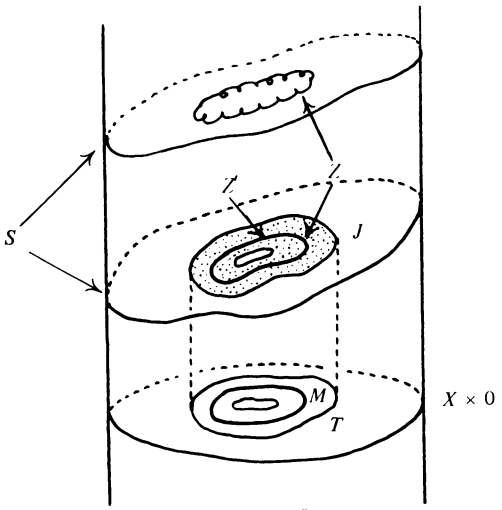

FIGURE 5

THEOREM 2.10 (MAIN THEOREM). *Let M be a closed smooth submanifold of a nonsingular algebraic set V with totally algebraic homology. Let M_i, $i = 1, 2, \ldots, k$, be closed smooth submanifolds of M in general position. Then there is an arbitrarily small isotopy of $V \times \mathbf{R}^n$ for some n, which simultaneously takes $M \times 0$ and all $M_i \times 0$ to nonsingular projectively closed algebraic sets.*

REMARK. By letting $V = \mathbf{R}^m$ we get Theorem 1.1 as a corollary to this theorem.

COROLLARY 2.11. *The interior of a compact smooth manifold M with $\partial M \neq \emptyset$, is diffeomorphic to a nonsingular algebraic set V which is properly imbedded in $\mathbf{R}^n$ for some n.*

PROOF. By $[\mathbf{AK}_2]$ or $[\mathbf{AK}_3]$ we can find closed smooth submanifolds M_i, $i = 1, \ldots, k$, of a manifold M' in general position with $\partial M' = \partial M$ and $M' / \cup M_i \simeq \operatorname{cone}(\partial M')$. Let $W = M \cup_\partial M'$. By Theorem 2.10 we can assume that W is a nonsingular algebraic set and all M_i are nonsingular algebraic subsets in some $\mathbf{R}^p$. Let $f(x)$ and $g(x)$ be polynomials with $f^{-1}(0) = W$, $g^{-1}(0) = \cup M_i$. Let $F(x, t) = f(x)^2 + (tg(x) - 1)^2$. Then let $F^{-1}(0) = V \subset \mathbf{R}^n$, $n = p + 1$. The algebraic set V has the required properties.

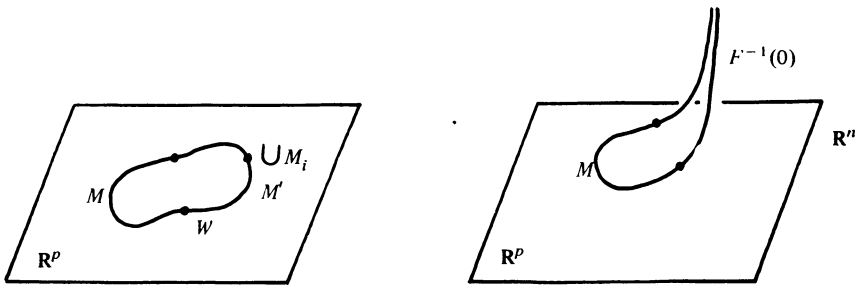

FIGURE 6

3. A quick proof of a special case. We can try to prove Theorem 2.10 by first assuming M to be a nonsingular algebraic subset of $V \times \mathbf{R}^m$ for some m (by Lemma 2.9′) and then isotoping the submanifolds M_i to nonsingular algebraic subsets of M. This seems to be very difficult at the present time; but if we assume that M has totally algebraic homology then we can achieve this Theorem 3.1. Theorem 3.1 has also a somewhat stronger conclusion than Theorem 2.10, also in Theorem 3.1 M can be noncompact.

THEOREM 3.1. *Let M be a nonsingular algebraic set with totally algebraic homology, and let M_i, $i = 1, \ldots, k$, be closed smooth submanifolds of M in general position. Then there is an arbitrarily small isotopy of $M \times \mathbf{R}^n$ for some n, which simultaneously takes $M \times 0$ and all $M_i \times 0$ to nonsingular algebraic sets. In addition, this isotopy takes $M \times 0$ to an algebraic set with totally algebraic homology.*

PROOF. We prove this by induction on k. Assume that $M_1, M_2, \ldots, M_{k-1}$ are already nonsingular algebraic subsets of M. Since M has totally algebraic

homology by Lemma 2.9′ M_k is isotopic to a nonsingular algebraic subset Z_k of $M \times \mathbf{R}^n$ by an arbitrarily small isotopy, for some n. By extending this isotopy we can assume that it takes $M \times 0$ to some manifold M' in $M \times \mathbf{R}^n$ such that $M' - M \times 0$ has compact closure and the composition $M' \hookrightarrow M \times \mathbf{R}^n \to M$ is a diffeomorphism. (In particular $Z_k \subset M'$.)

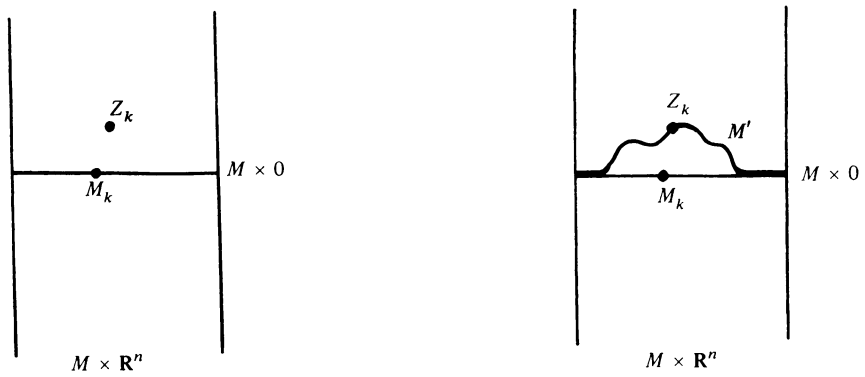

FIGURE 7

By Lemma 2.4 we may pick a proper polynomial ρ: $M \times \mathbf{R}^n \to \mathbf{R}$ so that $\rho^{-1}(0) = Z_k$. Let α: $M \to \mathbf{R}^n$ be a smooth function with compact support so that $M' = \{(x, y) \in M \times \mathbf{R}^n | y = \alpha(x)\}$. Pick a smooth function β: $M \times \mathbf{R}^n \to [0, 1]$ with compact support with $\beta(x, y) = 1$ when $|y| < |\alpha(x)|$. We now define γ: $M \times \mathbf{R}^n \to \mathbf{R}^n$:

$$\gamma(x, y) = \beta(x, y)(y - \alpha(x)) + (1 - \beta(x, y))\rho^2(x, y)y.$$

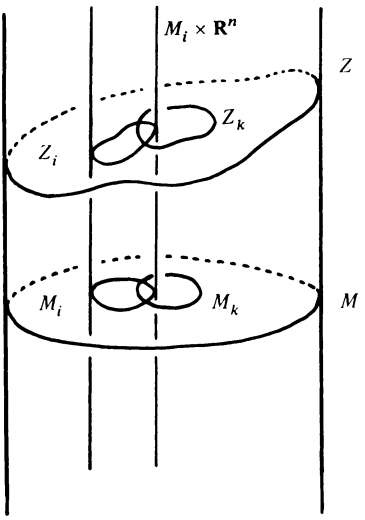

FIGURE 8

By inspection $\gamma^{-1}(0) = \{(x, y) \in M \times \mathbf{R}^n | y = \alpha(x)\} = M'$ and γ equals the polynomial $\rho^2(x, y)y$ outside of a compact set. By Lemma 2.5 we can C^1 approximate $\gamma(x, y) - \rho^2(x, y)y$ by an entire rational function u: $(M \times \mathbf{R}^n, Z_k) \to (\mathbf{R}^n, 0)$. This approximation is close on all $M \times \mathbf{R}^n$, not just on a compact set. Hence if we let $w(x, y) = u(x, y) + \rho^2(x, y)y$, then w is C^1 close to γ on $M \times \mathbf{R}^n$. Then the properness of $\rho(x, y)$ insures that $w^{-1}(0)$ is isotopic to $\gamma^{-1}(0) = M'$ by an arbitrarily small isotopy fixing Z_k. In particular the nonsingular algebraic set $Z = w^{-1}(0)$ contains Z_k. Let $Z_i = Z \cap (M_i \times \mathbf{R}^n)$, $i = 1, \ldots, k - 1$. By transversality each Z_i is a nonsingular algebraic subset of Z which is isotopic to M_i.

By Lemma 2.1 we have an isotopy of M carrying M_k to M_k' and fixing M_i for $i < k$, where M_k' is the image of Z_k under the projection $Z_k \subset M \times \mathbf{R}^n \to M$. We follow this isotopy in the $\mathbf{R}^n$ coordinate to isotop $(M, \{M_i\})$ to $(Z, \{Z_i\})$.

To finish the theorem we must prove that Z has totally algebraic homology. Take any $\alpha \in H_i(Z; \mathbf{Z}/2\mathbf{Z})$. Let h: $Z \to M$ be the diffeomorphism induced by projection. Pick an i-dimensional nonsingular algebraic set $U \subset M \times \mathbf{R}^k$ so that $\pi_*([U]) = h_*(\alpha)$, where π: $U \to M$ is induced by projection $M \times \mathbf{R}^k \to M$ and $[U]$ is the fundamental class of U. Let $V \subset Z \times \mathbf{R}^k$ be the pullback $V = \{(x, y, t) \in M \times \mathbf{R}^n \times \mathbf{R}^k | (x, y) \in Z, (x, t) \in U\}$. Then V is diffeomorphic to U and $\pi'_*([V]) = \alpha$ where π': $V \to Z$ is induced by the projection $M \times \mathbf{R}^n \times \mathbf{R}^k \to M \times \mathbf{R}^n$. Hence Z has totally algebraic homology.

4. Proof of Main Theorem. The following lemma basically says that for any given closed smooth manifold M and transversally intersecting closed smooth submanifolds M_i, $i = 1, \ldots, k$, then there is a nonsingular algebraic set Z and transversally intersecting nonsingular algebraic subsets Z_i, $i = 1, \ldots, k$, such that $(M; M_1, \ldots, M_k)$ is relatively cobordant to $(Z; Z_1, \ldots, Z_k)$. In other words, this lemma proves Theorem 2.10 up to cobordism. Then the proof of Theorem 2.10 is done by showing this "cobordism" implies the "isotopy". The idea of the proof of this lemma is similar to that of [**A**], where it was shown that for some P.L. manifolds M (M, the nonsmooth points of M) is relatively cobordant to (an algebraic set V, Sing V).

LEMMA 4.1. *Let M be a closed smooth manifold, and let M_i, $i = 1, 2, \ldots, k$, be closed smooth submanifolds of M in general position. Let φ be a map φ: $M \to V$ where V is a nonsingular algebraic set with totally algebraic homology. Then there is a smooth manifold T with submanifolds T_i in general position, and a map ψ: $T \to V$ with the following properties*:

(1) $\partial T \cap T_i = \partial T_i$.

(2) *∂T is the disjoint union $M \cup \bigcup_I U^I$ where the U^I are projectively closed nonsingular algebraic subsets of $V \times R^n$ for some n and I runs over all subsets of* $\{1, 2, \ldots, k\}$.

(3) $\psi|_M = \varphi$ *and* $\psi|_{U^I}$ *is induced by projection* $V \times \mathbf{R}^n \to V$.

(4) $T_i \cap M = M_i$.

(5) $\bigcap_{i \in J} T_i \cap U^I$ *is empty if* $J \not\subset I$ *and is a nonsingular algebraic set* U^I_J *if* $J \subset I$.

(6) *Let c_i by the codimension of M_i in M. Then there is a large m and entire rational functions λ_i^I: $U_I^I \to G(m, c_i)$ so that for $J \subset I$ $U_J^I = \{(x, y) \in U_I^I \times R^{(m+1)k} | (\lambda_i^I(x), y_i) \in E^*(m, c_i)$ if $i \in I - J$, and $y_i = 0$ if $i \notin I - J\}$. (We think of $R^{(m+1)k}$ as $(m+1) \times k$ matrices then y_i is the ith column vector of y.) Thus $U^I = U_\phi^I$ is the total space of a c_i sphere bundle over $U^I \cap T_i = U_i^I$ for each $i \in I$. Notice that we may extend each λ_i^I to U_i^I by composing with projection of U_i^I to U_I^I. This λ_i^I classifies the normal bundle of U_i^I in U^I.*

(7) *U^I is empty if $\cap_{i \in I} M_i$ is empty.*

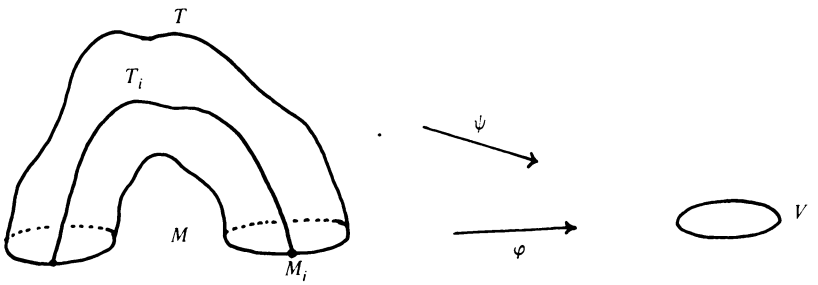

FIGURE 9

PROOF. After the end of this proof we give a few examples of the construction of T and the T_i's.

For any $I \subset \{1, \ldots, k\}$ let $M_I = \cap_{i \in I} M_i$. By convention, $M_\phi = M$.

We will prove this by induction on the number of subsets I of $\{1, \ldots, k\}$ so that M_I is nonempty. To start the induction, if M_I is empty for all subsets then $M = M_\phi$ is empty so the claim is true, letting T be the empty manifold.

Now take any M and M_i as above. Pick a subset I_0 of $\{1, \ldots, k\}$ so that M_{I_0} is nonempty but M_I is empty for all $I \supsetneq I_0$. Denote $L = M_{I_0}$. Pick m large enough that $G(m, c)$ is a classifying space for all c bundles over manifolds of dimension $\leqslant \dim M - c$. For each i we have a map α_i: $M_i \to G(m, c_i)$ classifying the normal bundle of M_i in M. We have a map β: $L \to V \times \prod_{i \in I_0} G(m, c_i)$ which is $\varphi|_L$ to V and $\alpha_i|_L$ to $G(m, c_i)$. By Lemma 2.8 there is a manifold K and a map γ: $K \to V \times \prod_{I_0} G(m, c_i)$ so that ∂K is the disjoint union of L and a projectively closed nonsingular algebraic set $U' \subset V \times \prod_{I_0} G(m, c_i) \times R^a$, $\gamma|_L = \beta$ and $\gamma|_{U'}$ is the obvious projection. Define μ_j: $V \times \prod G(m, c_i) \to G(m, c_j)$ to be projection. Define manifolds S_I by letting S_I be empty if $I \not\subset I_0$, and if $I \subset I_0$

$$S_I = \{(x, y) \in K \times R^{(m+1)k} | (\mu_i \gamma(x), y_i) \in E^*(m, c_i) \quad \text{if } i \in I_0 - I, \text{ and } y_i = 0 \text{ if } i \notin I_0 - I\}$$

where $y_i \in \mathbf{R}^{m+1}$ is the ith coordinate of y. Since $\mu_i \gamma|_L = \alpha_i|_L$ there is a diffeomorphism h: $A \to B$ where A is a closed tubular neighborhood of $L \times 0$ in ∂S_ϕ and B is a tubular neighborhood of L in M. Furthermore, $h(A \cap S_i) = B \cap M_i$ for each $i \in I_0$ and $B \cap M_i$ is empty for each $i \notin I_0$. We now form a manifold $S' = S_\phi \cup M \times [0, 1]$ where we identify $A \subset \partial S_\phi$ with $B \times 1 \subset M \times [0, 1]$ via h. Let $S_I' = S_I \cup M_I \times [0, 1] \subset S'$. Then the S_i' are in general position in S' and $\cap_{i \in I} S_i' = S_I'$. We may define a map θ: $S' \to V$ by first retracting S' to $S_\phi \cup L \times [0, 1] \cup M \times 0$

and then mapping S_ϕ by projecting to K, mapping by γ to $V \times \coprod G(m, c_i)$ and projecting to V. The map on $L \times [0, 1] \cup M \times 0$ will be projection to M followed by φ.

Notice that $\theta|_{M\times 0} = \varphi$. Also if we denote $U^{I_0} = S_\phi \cap U' \times R^{(m+1)k}$ then $\theta|_{U^{I_0}}$ is projection. In addition $\partial S'$ is the disjoint union of $M \times 0$, U^{I_0} and $M' = (S_\phi \cap L \times R^{(m+1)k} - \mathring{A}) \cup (M - \mathring{B}) \times 1$. Let $M'_I = M' \cap S'_I$. Then M'_I is empty if M_I is. Also M'_{I_0} is empty. So by induction there are T', T'_i, and ψ': $T' \to V$ satisfying (1)–(6) (with everything primed and $\psi'|_{M'} = \theta_{M'}$). Let $T = S' \cup T'$, $T_i = S'_i \cup T'_i$ and $\psi = \psi' \cup \theta$ then we are done.

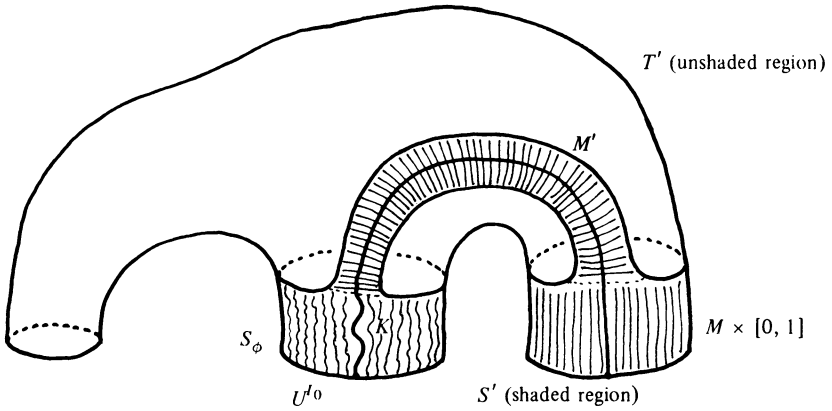

FIGURE 10

As promised, we now give examples of the T and T_i's constructed in the above proof. We let V be a point so we do not have to mention the maps φ or ψ.

EXAMPLE A. $M = S^1$, $k = 1$, M_1 is a point. Then $L = M_1$ and K can be an interval. U^1 is a circle and U_1^1 is a point. Figure 11(a) below shows S'. Notice that $M' \cap S'_1$ is empty. Now since M' bounds we may let U^Φ be empty, then Figure 11(b) shows T.

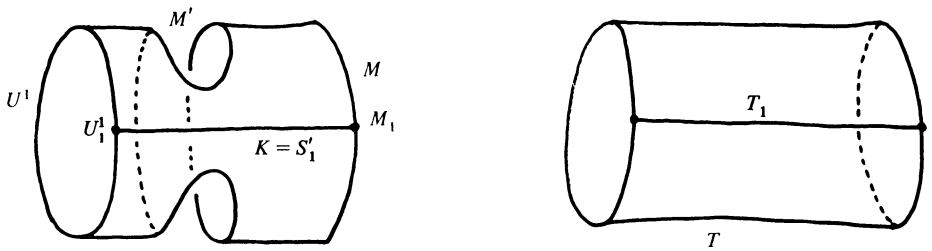

FIGURE 11

EXAMPLE B. Let $M = RP^2$ and let M_1 and M_2 be linear subspaces $M_i = \{(x_0 : x_1 : x_2) | x_i = 0\}$. Then M_{12} is a point. We let k be an interval. Then U^{12} is a torus $S^1 \times S^1$ and $U_1^{12} = S^1 \times b$ and $U_2^{12} = a \times S^1$. M' is the connected sum of a torus and a projective plane. (In fact, M' is the torus $S^1 \times S^1$ blown up at the point (a, b), the M'_i are the blowups of the $S^1 \times b$ and $a \times S^1$ and are disjoint.)

By representing M' as a hexagon with sides identified we indicate M_1' and M_2' in Figure 12. Now the normal bundle of each M_i' in M' is nontrivial, hence we let U^i be the nontrivial S^1 bundle over S^1, i.e., the U^i are Klein bottles and the U_i^i are circles, the zero sections of these bundles. After glueing these on we are left with an M'' which is in fact $S^1 \times S^1 \# RP^2$. Also the M_i'' are all empty. We cobord the M'' to RP^2, say and let $U^\Phi = RP^2$.

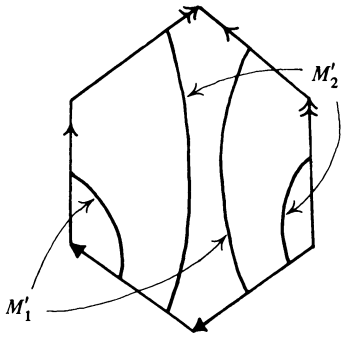

FIGURE 12

EXAMPLE C. $M = S^2$, M_1 and M_2 are two great circles and M_3 is a point in $S^2 - (M_1 \cup M_2)$. Then U^{123}, U^{13} and U^{23} must be empty. We may let U^3 be a two sphere and U_3^3 a point in U^3. S' is a solid disc with two holes and M' is a two sphere and M_1' and M_2' are circles (see Figure 13). Since $M_1' \cap M_2'$ is two points, it bounds so we may make U^{12} empty. We thus add an $S^1 \times S^1 \times [-1, 1]$ to our cobordism. Our new M'' is the connected sum of two tori and M_1'' and M_2'' are as in Figure 14.

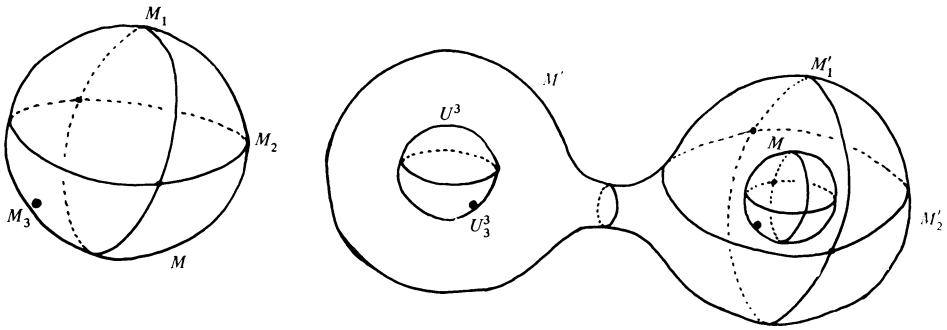

FIGURE 13

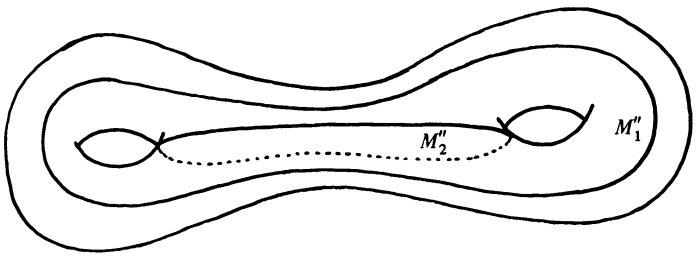

FIGURE 14

Since M_1'' and M_2'' bound we may make U^1 and U^2 be empty. We are then left with a surface of genus 2 for M''' which bounds so we may let U^Φ be empty also. The actual T we have constructed is rather hard to describe, it is the union of two $S^2 \times [0, 1]$'s, an $S^1 \times S^1 \times [0, 1]$, two $D^2 \times S^1$'s and a manifold with boundary $S^1 \times S^1 \# S^1 \times S^1$. The T_i's are easier to describe. T_3 is an interval. T_1 and T_2 are each a torus minus a disc. $T_1 \cap T_2$ is an interval (see Figure 15).

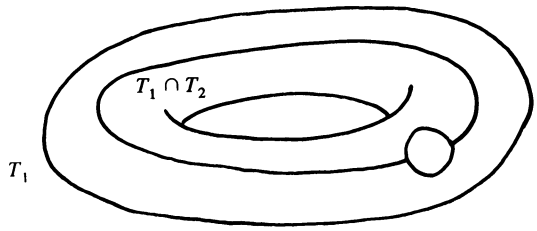

FIGURE 15

PROOF OF THEOREM 2.10. Before proceeding with a detailed proof, we give a rough sketch to guide the reader. Using Lemma 4.1 we find a closed smooth submanifold $N \subset V \times \mathbf{R}^n$ for some n (N is essentially the double of T of Lemma 4.1), and submanifolds $N_i \subset N$, $i = 1, \ldots, k$, with the properties that:

(1) M is a smooth submanifold of N.

(2) There is a projectively closed nonsingular algebraic set $U \subset V \times \mathbf{R}^n$ with $U \subset N$ and disjoint from M.

(3) $M \cup U$ is a codimension 1 submanifold of N, which compactly separates N.

(4) $N_i \cap M = M_i$.

(5) $N_i \cap U$ is a nonsingular algebraic set. There is also a technical condition which will appear in the proof, roughly it says that for each i the connected components of U which intersect N_i are the total space of an "algebraic" sphere bundle over $N_i \cap U$.

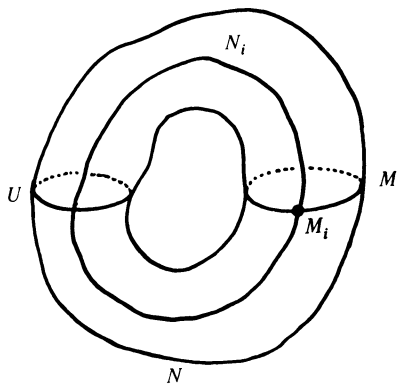

FIGURE 16

We then proceed to make N, M and N_i's algebraic. We show that for each i there are algebraic sets Y_i and Y_i' so that the pair (N, N_i) is diffeomorphic to (Y_i, Y_i') by a diffeomorphism σ_i so that $\sigma_i|_U$ is an entire rational function. By Lemma 2.9 we may isotop N to a nonsingular component Z' of an algebraic set Z fixing U so that all the above diffeomorphisms to (Y_i, Y_i') are entire rational functions. Hence N_i have become algebraic by Lemma 2.2. Now by Lemmas 2.6 and 2.3 we can isotop M to a nonsingular algebraic subset X of Z'. Hence each M_i is isotopic to the nonsingular algebraic set $X_i = N_i \cap X$.

We now proceed with the proof: Let φ: $M \to V$ be the inclusion. Let T be the cobordism obtained in Lemma 4.1. Then by making sure n is big enough, we get an imbedding η: $T \to V \times R^n$ so that $\eta(x) = (x, 0)$ for $x \in M$ and $\eta|_{U^I}$ is inclusion for each I. (It might be necessary to translate the U^I's a bit to make them disjoint.)

We now let N be the boundary of $\eta(T) \times [-1, 1] \subset (V \times R^n) \times R$ after smoothing out the corners. Also $N_i \subset N$ are the boundaries of $\eta(T_i) \times [-1, 1]$, $i = 1, \ldots, k$. Thus N is the double of T and the N_i are the doubles of T_i. Denote $U = \bigcup_{I\{1,\ldots,k\}} U^I \times 0$. Then $M \times 0 \times 0$, U and the N_i's are in general position in N, $M \times 0 \times 0 \cup U$ compactly separates N and the germ of N at U is the germ of the nonsingular algebraic set $\bigcup_I U^I \times R$ at U.

We now construct the Y_i and Y_i'. Pick β_i: $N_i \to G(m, c_i)$ classifying the normal bundle of N_i in N. We may require that $\beta_i|_{N_i \cap U}$ be an entire rational function and that in fact $\beta_i|_{N_i \cap U^I} = \beta_i|_{U_i^I}$ is the composition of projection to U_I^I and λ_i^I if $i \in I$ (see Lemma 4.1). By Lemma 2.9′ there is a nonsingular algebraic set $Y_i' \subset V \times R^n \times R \times R^a$, an entire rational function γ_i: $Y_i' \to G(m, c_i)$ and a diffeomorphism h_i: $N_i \to Y_i'$ so that $h_i(x) = (x, 0)$ and $\gamma_i(x, 0) = \beta_i(x)$ for all $x \in N_i \cap U$. In addition, $\gamma_i h_i$ approximates β_i, hence $\gamma_i h_i$ and β_i are homotopic. Define $Y_i'' = \{(x, y) \in Y_i \times R^{m+1} | (\gamma_i(x), y) \in E^*(m, c_i)\}$. Then since $\gamma_i h_i$ and β_i are homotopic, the normal bundle of Y_i' in Y_i'' is equivalent to the normal bundle of N_i in N. Hence there is an imbedding σ_i: $N \to V \times R^n \times R \times R^a \times R^{m+1}$ so that $\sigma_i|_U$ is an entire rational function, $\sigma_i(U) \subset Y_i''$, $\sigma_i|_{N_i} = h_i$ and the germ of $\sigma_i(N)$ at $\sigma_i(U \cup N_i)$ is the germ of Y_i'' at $\sigma_i(U \cup N_i)$. Here we use the special properties of U. For instance we could construct σ_i by letting $\sigma_i(x, 0) = (x, 0, 0, \pi_i(x))$ for all $x \in U^I \subset V \times R^n$ and letting $\sigma_i|_{N_i} = h_i$. We may extend this σ_i to a neighborhood of $U \cup N_i$. Using the fact that there is an isotopy of $V \times R^n \times R \times R^a \times R^{m+1}$ whose time one map restricted to $N_i \times 0 \times 0$ is h_i, we may extend σ_i to all of N. Now by Lemma 2.9′ we may isotop $\sigma_i(N)$ to a nonsingular algebraic set fixing $\sigma_i(U \cup N_i) = \sigma_i(U) \cup Y_i'$. Hence we may assume in fact that $\sigma_i(N)$ is a nonsingular algebraic set Y_i.

Now by applying Lemma 2.9 to the diagonal map $N \to Y_1 \times Y_2 \times \cdots \times Y_k$ (recall $N \approx Y_i$) there is an integer b, a small isotopy h_t' of $V \times R^n \times R \times R^b$ fixing $U \times 0$, an algebraic set $Z \subset V \times R^n \times R \times R^b$ and an entire rational function $u = (u_1, \ldots, u_k)$: $Z \to Y_1 \times Y_2 \times \cdots \times Y_k$ so that $h_1'(N \times 0) = Z'$ is a nonsingular component of Z, $u_i(x, 0) = \sigma_i(x)$ for all $x \in U$, and the $u_i h_1'$ are closed approximations to the σ_i, $i = 1, \ldots, k$. By Lemma 2.6 there is a small isotopy in Z' taking $U \times 0 \cup h_1'(M \times 0 \times 0 \times 0)$ to a nonsingular projectively closed algebraic subset X' of Z so that $U \times 0 \subset X'$. But then by Lemma 2.3 $X = X' - U \times 0$ is a projectively closed nonsingular algebraic set.

The fact that the $u_i h_1'$ are close approximations to σ_i means that there is a small isotopy from each $u_i^{-1}(Y_i') \cap Z'$ to $h_1'(N_i \times 0) = h_1' \sigma_i^{-1}(Y_i')$. Also $u_i^{-1}(Y_i') \cap Z'$ consists of nonsingular points of dimension dim $N - c_i$ by Lemma 2.2. Hence by Lemma 2.1 there is a small ambient isotopy h_t'' of Z' so that $h_1'' h_1'(N_i \times 0) = u_i^{-1}(Y_i') \cap Z'$ and $h_1'' h_1'(M \times 0 \times 0 \times 0) = X$. Let $X_i = u_i^{-1}(Y_i') \cap X$. Notice that X_i are nonsingular since they are the transverse intersections of nonsingular components of algebraic sets. Also $X_i = h_1'' h_1'(M_i \times 0 \times 0 \times 0)$. Therefore we have proven the theorem.

REFERENCES

[A] S. Akbulut, *Algebraic equations for a class of P.L. spaces*, Math. Ann. **231** (1977), 19–31.

[AK_1] S. Akbulut and H. King, *Real algebraic variety structures of P.L. manifolds*, Bull. Amer. Math. Soc. **83** (1977), 281–282.

[AK_2] _____, *Topology of real algebraic sets with isolated singularities*, Ann. of Math. **113** (1981).

[AK_3] _____, *All knots are algebraic*, Comment. Math. Helv. **56** (1981).

[CF] P. Conner and E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 33, Springer-Verlag, Berlin, 1964.

[H] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1964), 109–326.

[L] S. Lojasiewicz, *Sur le problème de la division*, Rozprawy Math. **22** (1961), 1–57.

[M] J. Milnor, *On the Steifel-Whitney numbers of complex manifolds and of spin manifolds*, Topology **3** (1965), 223–230.

[N] J. Nash, *Real algebraic manifolds*, Ann. of Math. **56** (1952), 405–421.

[T] A. Tognoli, *Su una congettura di Nash*, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. **27** (1973), 167–185.

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